Indian Statistical Institute, Bangalore

B. Math.(hons.), Third Year, First Semester

Probability-III

Mid Term Examination | Date : 12 September 2024

Maximum marks: 20 | Time: 2 hours

Answer any 4, each question carries 5 marks.

Kindly ensure your writing is clear and if you are using any results please provide as many details as you can.

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\widetilde{\mathcal{F}} \equiv \{A : B_1 \subset A \subset B_2 \textit{ for some } B_1, B_2 \in \mathcal{F} \textit{ satisfying } \mu(B_2 \setminus B_1) = 0\}$. For any $A \in \widetilde{\mathcal{F}}$, set

$$\widetilde{\mu}(A) = \mu(B_1) = \mu(B_2)$$

for any pair of sets $B_1, B_2 \in \mathcal{F}$ with $B_1 \subset A \subset B_2$ and $\mu(B_2 \setminus B_1) = 0$. Then,

   (i) $\widetilde{\mathcal{F}}$ is a σ-algebra with $\mathcal{F} \subset \widetilde{\mathcal{F}}$, and $\widetilde{\mu}$ is well defined,

   (ii) $(\Omega, \widetilde{\mathcal{F}}, \widetilde{\mu})$ is a complete measure space and $\widetilde{\mu} = \mu$ on $\mathcal{F}$.

   (Recall that a measure space $(\Omega, \mathcal{F}, \mu)$ is called complete if for any $A \in \mathcal{F}$ with $\mu(A) = 0$, and for any $B \subset A$, it follows that $B \in \mathcal{F}$. In other words, if A is a null set, then all subsets of A are also in $\mathcal{F}$.

2. Prove or disprove the following statements:

   (i) If $\mathcal{F}$ is a family of subsets of a set Ω and if A is an element of $\sigma(\mathcal{F})$, then there exists a countable subset C of $\mathcal{F}$ such that $A \in \sigma(C)$.

   (ii) If $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a sequence of σ-algebras on a given set such that for each $n \in \mathbb{N}$, $\mathcal{F}_n$ is a proper subset of $\mathcal{F}_{n+1}$, then $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is always a σ-algebra on the given set.

3. Let X be a random variable such that $M_X(t) \equiv \mathbb{E}(e^{tX}) < \infty$ for $|t| < \delta$ for some $\delta > 0$.

   (i) Show that $\mathbb{E}(e^{tX}|X|^r) < \infty$ for all $r > 0$ and $|t| < \delta$.

   (ii) Show that $M_X^{(r)}(t)$, the r-th derivative of $M_X(t)$ for $r \in \mathbb{N}$, satisfies

$$M_X^{(r)}(t) = \mathbb{E}(e^{tX} X^r) \textit{ for } |t| < \delta.$$

4. Let $\{X_n\}_{n\geq1}$ be a sequence of random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) If $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varepsilon) < \infty$ for each $\varepsilon > 0$, then $\mathbb{P}\left(\lim_{n\to\infty} X_n = 0\right) = 1$.

(ii) If $\{X_n\}_{n\geq 1}$ are pairwise independent and $\mathbb{P}\left(\lim_{n\to\infty} X_n = 0\right) = 1$, then $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varepsilon) < \infty$ for each $\varepsilon > 0$.

5. Let $\overline{\mathbb{R}}$ be extended real number and $\mathcal{T}$ be the tail σ-algebra of a sequence of independent random variables $\{X_n\}_{n\geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ (i.e. $\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma\left(\{X_j : j \geq n\}\right)$). Then, prove the following results:

   (i) Let X be a $\langle \mathcal{T}, \mathcal{B}(\overline{\mathbb{R}})\rangle$-measurable $\overline{\mathbb{R}}$-valued random variable from Ω to $\overline{\mathbb{R}}$. Then, there exists $c \in \overline{\mathbb{R}}$ such that

   $$\mathbb{P}(X = c) = 1.$$

   (ii) Let $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^2] = 1$ for all $n \geq 1$. Let $S_n = X_1 + \cdots + X_n$, $n \geq 1$, and $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy$, $x \in \mathbb{R}$. If $\mathbb{P}(S_n \leq \sqrt{n}x) \to \Phi(x)$ for all $x \in \mathbb{R}$, then

   $$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = +\infty \ a.s.$$